The Atmospheric Continuum

- In atmospheric dynamics (ocean dynamics also) we do not treat the fluid as a collection of individual molecules.
- Instead we treat the fluid as a continuous medium (or continuum) in which a “point” is a volume element that is very small compared to the total fluid volume but still contains a very large number of molecules.
- These volume elements are commonly called “air parcels” or “air particles.”
- The properties of these volume elements describe the state of the atmosphere.
- For further reading, see Martin section 1.1.

Newton’s Second Law

- Atmospheric motions are governed by Newton’s second law of motion, which states that the rate of change of momentum of an object equals the sum of all the forces acting.

\[
\frac{d}{dt} (mv) = \sum F
\]

\[
m \frac{dv}{dt} = \sum F
\]

\[
a = \sum F
\]
Fundamental Forces

• To understand atmospheric motions, Newton’s second law requires that we understand the forces that act on a fluid parcel.
• There are two types of forces: body forces and surface forces.
• Body forces act on the center of mass of fluid parcel and have magnitudes proportional to the mass of the parcel.
• Surface forces act across the boundary surface separating a fluid parcel from its surroundings and have magnitudes independent of the mass of the parcel.
• For further reading, see Martin section 2.1.

Pressure Gradient Force

We consider a very small volume element of air
\[ \delta V = \delta x \delta y \delta z \]
that is centered at the point \((x_0, y_0, z_0)\)
Pressure force exerted on left wall: $F_{Bx}$

Pressure force exerted on right wall: $F_{Ax}$

Net x component of pressure force:
$F_x = F_{Ax} + F_{Bx}$

Method: Use Taylor expansion to develop a mathematical expression for the pressure force at the center of this fluid element.

Taylor series expansion:
\[ f(x) = f(x_0) + f'(x_0)(x-x_0) + \text{higher order terms} \]

In this case (neglecting higher order terms):
\[ p(x) = p(x_0) + \frac{\partial p}{\partial x} (x-x_0) \]

Therefore, we can express the pressure forces as
\[ F_{Ax} = -\left(p_0 + \frac{\partial p}{\partial x} \frac{\partial x}{2}\right) \delta y \delta z \]
\[ F_{Bx} = +\left(p_0 - \frac{\partial p}{\partial x} \frac{\partial x}{2}\right) \delta y \delta z \]
In the same manner, it can be shown that

\[
F_x = -\frac{1}{\rho} \frac{\partial p}{\partial x}
\]

\[
F_y = -\frac{1}{\rho} \frac{\partial p}{\partial y}
\]

\[
F_z = -\frac{1}{\rho} \frac{\partial p}{\partial z}
\]

In vector form:

\[
\vec{F} = -\frac{1}{\rho} \nabla p
\]

Note: Pressure force is proportional to gradient of pressure.
Newton’s law of universal gravitation states that gravitational force exerted by mass $M$ on mass $m$ is:

$$\vec{F}_g = -\frac{GMm}{r^2} \vec{\frac{r}{r}}$$

If the earth is taken to be the mass $M$ and $m$ is taken to be the mass of a fluid parcel or volume element, then we can write the force per unit mass exerted on the fluid by the earth as

$$\frac{\vec{F}_g}{m} = \vec{g}^* = -\frac{GM}{r^2} \vec{\frac{r}{r}}$$

If $a$ is the radius of the earth and $z$ is the distance above sea level, then

$$\vec{g}^* = \frac{\vec{g}_0^*}{(1 + z/a)^2}, \quad \text{where} \quad \vec{g}_0^* = \left(\frac{GM}{a^2}\right) \vec{\frac{r}{r}}$$

Because the depths of the atmosphere and ocean are small compared to the radius of the earth ($z << a$) we can treat the gravitational force per unit mass as a constant.

$$\vec{g}^* = \vec{g}_0^*$$
Viscous Force

- If the wind velocity varies with height, random molecular motions will cause momentum to be transferred vertically.
- In other words, there is a drag exerted by the layers above and below the level of interest.

\[ \text{The stress due to the velocity shear is given by} \]
\[ \tau_x = \mu \frac{\partial u}{\partial z} \]

where \( \mu \) is the dynamic viscosity coefficient.

Using Taylor series expansion to express the net viscous force:

\[ \left( \tau_{xx} + \frac{\partial \tau_{xx}}{\partial z} \frac{\partial \xi}{\partial z} \right) \partial y \partial z = \left( \tau_{xx} + \frac{\partial \tau_{xx}}{\partial z} \frac{\partial \xi}{\partial z} \right) \partial y \partial z \]
\[ = \frac{\partial \tau_{xx}}{\partial z} \partial x \partial y \partial z \]
Net viscous force \( \frac{\partial \tau_{zz}}{\partial z} \) \( \frac{\partial \partial \partial \delta \partial \delta \delta}{\partial \partial \partial \delta \partial \delta \delta} \)

Dividing the above expression by the mass \( \rho \) \( \partial \partial \partial \delta \partial \delta \delta \) yields the viscous force per unit mass:

\[
\frac{1}{\rho} \frac{\partial \tau_{zz}}{\partial z} - \frac{1}{\rho} \frac{\partial}{\partial z} \left( \mu \frac{\partial u}{\partial z} \right) = \nu \frac{\partial^2 u}{\partial z^2}
\]

\( \nu = \frac{\mu}{\rho} = \) kinematic viscosity coefficient = \(1.46 \times 10^{-5} \text{ m}^2 \text{s}^{-1}\)

Molecular viscosity is too small to be important except very close (cm) to the Earth's surface and above 100 km. Other sources of momentum transfer are more important in the lower atmosphere, and these will be discussed later in the semester.