

# The Vorticity Equation

To understand the processes that produce changes in vorticity, we would like to derive an expression that includes the time derivative of vorticity:

$$\frac{d}{dt} \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) = \dots$$

Recall that the momentum equations are of the form

$$\frac{du}{dt} = \dots \quad \text{x-component momentum equation}$$

$$\frac{dv}{dt} = \dots \quad \text{y-component momentum equation}$$

Thus we will begin our derivation by taking

$$\frac{\partial}{\partial x} [\text{y-component momentum equation}] - \frac{\partial}{\partial y} [\text{x-component momentum equation}]$$

$$\begin{aligned} & \frac{\partial}{\partial x} [\text{y-component momentum equation}] - \frac{\partial}{\partial y} [\text{x-component momentum equation}] = \\ & \frac{\partial}{\partial x} \left[ \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} + fu = -\frac{1}{\rho} \frac{\partial p}{\partial y} \right] - \frac{\partial}{\partial y} \left[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} - fv = -\frac{1}{\rho} \frac{\partial p}{\partial x} \right] \\ & \frac{\partial}{\partial x} \frac{\partial v}{\partial t} + u \frac{\partial^2 v}{\partial x^2} + \frac{\partial v}{\partial x} \frac{\partial u}{\partial x} + v \frac{\partial^2 v}{\partial x \partial y} + \frac{\partial v}{\partial y} \frac{\partial v}{\partial x} + w \frac{\partial^2 v}{\partial x \partial z} + \frac{\partial v}{\partial z} \frac{\partial w}{\partial x} + f \frac{\partial u}{\partial x} - \frac{1}{\rho} \frac{\partial^2 p}{\partial x \partial y} + \frac{1}{\rho^2} \left( \frac{\partial p}{\partial y} \frac{\partial \rho}{\partial x} \right) \\ & - \frac{\partial}{\partial y} \frac{\partial u}{\partial t} + u \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial u}{\partial x} \frac{\partial u}{\partial y} + v \frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} + w \frac{\partial^2 u}{\partial y \partial z} + \frac{\partial u}{\partial z} \frac{\partial w}{\partial y} - f \frac{\partial v}{\partial y} - v \frac{\partial f}{\partial y} = -\frac{1}{\rho} \frac{\partial^2 p}{\partial x \partial y} + \frac{1}{\rho^2} \left( \frac{\partial p}{\partial x} \frac{\partial \rho}{\partial y} \right) \\ & \frac{\partial}{\partial t} \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) + u \frac{\partial}{\partial x} \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) + v \frac{\partial}{\partial y} \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) + w \frac{\partial}{\partial z} \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) + \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + f \left( \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) \\ & + \left( \frac{\partial w}{\partial x} \frac{\partial v}{\partial z} - \frac{\partial w}{\partial y} \frac{\partial u}{\partial z} \right) + v \frac{\partial f}{\partial y} = \frac{1}{\rho^2} \left( \frac{\partial p}{\partial y} \frac{\partial \rho}{\partial x} - \frac{\partial p}{\partial x} \frac{\partial \rho}{\partial y} \right) \quad \frac{df}{dt} = \frac{\partial f}{\partial t} + u \frac{\partial f}{\partial x} + v \frac{\partial f}{\partial y} + w \frac{\partial f}{\partial z} \\ & \frac{\partial \zeta}{\partial t} + u \frac{\partial \zeta}{\partial x} + v \frac{\partial \zeta}{\partial y} + w \frac{\partial \zeta}{\partial z} + \zeta \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + f \left( \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) + \left( \frac{\partial w}{\partial x} \frac{\partial v}{\partial z} - \frac{\partial w}{\partial y} \frac{\partial u}{\partial z} \right) + v \frac{\partial f}{\partial y} = \frac{1}{\rho^2} \left( \frac{\partial p}{\partial y} \frac{\partial \rho}{\partial x} - \frac{\partial p}{\partial x} \frac{\partial \rho}{\partial y} \right) \\ & \frac{d}{dt} (\zeta + f) = -(\zeta + f) \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) - \left( \frac{\partial w}{\partial x} \frac{\partial v}{\partial z} - \frac{\partial w}{\partial y} \frac{\partial u}{\partial z} \right) + \frac{1}{\rho^2} \left( \frac{\partial p}{\partial y} \frac{\partial \rho}{\partial x} - \frac{\partial p}{\partial x} \frac{\partial \rho}{\partial y} \right) \quad \text{vorticity equation} \end{aligned}$$

## Terms In Vorticity Equation

$$\frac{d}{dt}(\zeta + f) = \underbrace{-(\zeta + f)}_A \left( \underbrace{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}}_B \right) - \underbrace{\left( \frac{\partial w}{\partial x} \frac{\partial v}{\partial z} - \frac{\partial w}{\partial y} \frac{\partial u}{\partial z} \right)}_C + \frac{1}{\rho^2} \underbrace{\left( \frac{\partial p}{\partial y} \frac{\partial \rho}{\partial x} - \frac{\partial p}{\partial x} \frac{\partial \rho}{\partial y} \right)}_D$$

- A: Rate of change of absolute vorticity following the fluid motion
- B: Effect of horizontal velocity divergence on vorticity
- C: Transfer of vorticity between horizontal and vertical components ("twisting term" or "tilting term")
- D: Effects of baroclinicity ("solenoidal term")

Term A: Rate of change of absolute vorticity following the fluid motion

$$\frac{d(\zeta + f)}{dt} = \frac{\partial(\zeta + f)}{\partial t} + u \frac{\partial(\zeta + f)}{\partial x} + v \frac{\partial(\zeta + f)}{\partial y} + w \frac{\partial(\zeta + f)}{\partial z}$$

local  
tendency  
of absolute  
vorticity

horizontal  
advection  
of absolute  
vorticity

vertical  
advection  
of absolute  
vorticity

Term B: Effect of horizontal velocity divergence on vorticity

$$-(\zeta + f) \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)$$

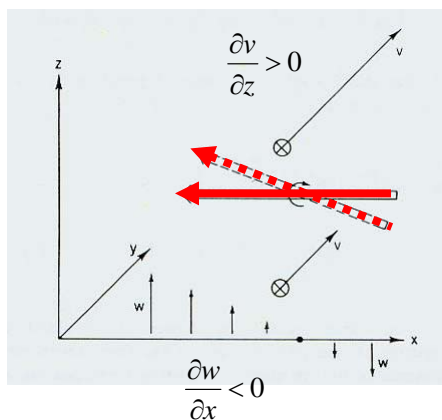
If  $\left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) > 0$  (divergence), then vorticity will decrease if absolute vorticity is positive. Vorticity will increase if absolute vorticity is negative.

If  $\left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) < 0$  (convergence), then vorticity will increase if absolute vorticity is positive. Vorticity will decrease if absolute vorticity is negative.

This mechanism is quite important for large-scale midlatitude systems.

Term C: Transfer of vorticity between horizontal and vertical components ("twisting term" or "tilting term")

$$-\left( \frac{\partial w}{\partial x} \frac{\partial v}{\partial z} - \frac{\partial w}{\partial y} \frac{\partial u}{\partial z} \right)$$



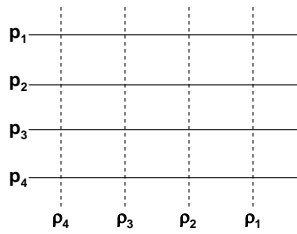
In this example, vertical shear of v-component wind is producing shear vorticity about an east-west axis. The orientation of the vorticity vector is shown by the solid red arrow.

East-west variations in the vertical velocity twist or tilt this "vortex tube" toward a more vertical orientation, as indicated by the broken red arrow. This gives the vorticity vector a component in the z-direction, indicating a transfer of vorticity from the horizontal to the vertical.

$$\frac{\partial v}{\partial z} \frac{\partial w}{\partial x} < 0 \rightarrow \frac{d(\zeta + f)}{dt} > 0$$

Term D: Effects of baroclinicity ("solenoidal term")

$$\frac{1}{\rho^2} \left( \frac{\partial p}{\partial y} \frac{\partial \rho}{\partial x} - \frac{\partial p}{\partial x} \frac{\partial \rho}{\partial y} \right)$$



$$p_4 > p_3 > p_2 > p_1$$

$$\rho_4 > \rho_3 > \rho_2 > \rho_1$$

This term arises because of the horizontal variations in density that occur in a baroclinic atmosphere. In this example, even though the pressure gradient is uniform, variations in density produce small variations in the pressure gradient force. The variations in acceleration that result lead to the production of positive vorticity.

$$\frac{\partial p}{\partial y} < 0; \frac{\partial \rho}{\partial x} < 0 \rightarrow \frac{d(\zeta + f)}{dt} > 0$$