

Vorticity Equation in Isobaric Coordinates

To obtain a version of the vorticity equation in pressure coordinates, we follow the same procedure as we used to obtain the z-coordinate version:

$$\frac{\partial}{\partial x}[\text{y-component momentum equation}] - \frac{\partial}{\partial y}[\text{x-component momentum equation}]$$

Using the p-coordinate form of the momentum equations, this is:

$$\frac{\partial}{\partial x} \left[\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + \omega \frac{\partial v}{\partial p} + fu = - \frac{\partial \Phi}{\partial y} \right] - \frac{\partial}{\partial y} \left[\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + \omega \frac{\partial u}{\partial p} - fv = - \frac{\partial \Phi}{\partial x} \right]$$

which yields:

$$\frac{d}{dt}(\zeta_p + f) = -(\zeta_p + f) \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)_p - \left(\frac{\partial \omega}{\partial y} \frac{\partial u}{\partial p} - \frac{\partial \omega}{\partial x} \frac{\partial v}{\partial p} \right)$$

vorticity equation in isobaric coordinates

Scaling The Vorticity Equation

Starting with the z-coordinate form of the vorticity equation, we begin by expanding the total derivative and retaining only nonzero terms:

$$\frac{\partial \zeta}{\partial t} + u \frac{\partial \zeta}{\partial x} + v \frac{\partial \zeta}{\partial y} + w \frac{\partial \zeta}{\partial z} + v \frac{\partial f}{\partial y} = -(\zeta + f) \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) - \left(\frac{\partial w}{\partial x} \frac{\partial v}{\partial z} - \frac{\partial w}{\partial y} \frac{\partial u}{\partial z} \right) + \frac{1}{\rho^2} \left(\frac{\partial p}{\partial y} \frac{\partial \rho}{\partial x} - \frac{\partial p}{\partial x} \frac{\partial \rho}{\partial y} \right)$$

Next, we will evaluate the order of magnitude of each of these terms. Our goal is to simplify the equation by retaining only those terms that are important for large-scale midlatitude weather systems.

Scaling Quantities

U = horizontal velocity scale
 W = vertical velocity scale
 L = length scale
 H = depth scale
 δP = horizontal pressure fluctuation
 ρ = mean density
 $\delta\rho/\rho$ = fractional density fluctuation
 T = time scale (advective) = L/U
 f_0 = Coriolis parameter
 β = "beta" parameter

Values of Scaling Quantities (midlatitude large-scale motions)

U	10 m s^{-1}
W	10^{-2} m s^{-1}
L	10^6 m
H	10^4 m
δP (horizontal)	10^3 Pa
ρ	1 kg m^{-3}
$\delta\rho/\rho$	10^{-2}
T	10^5 s
f_0	10^{-4} s^{-1}
β	$10^{-11} \text{ m}^{-1} \text{ s}^{-1}$

$$\frac{\partial \zeta}{\partial t} + u \frac{\partial \zeta}{\partial x} + v \frac{\partial \zeta}{\partial y} + w \frac{\partial \zeta}{\partial z} + v \frac{\partial f}{\partial y} = -(\zeta + f) \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) - \left(\frac{\partial w}{\partial x} \frac{\partial v}{\partial z} - \frac{\partial w}{\partial y} \frac{\partial u}{\partial z} \right) + \frac{1}{\rho^2} \left(\frac{\partial p}{\partial y} \frac{\partial \rho}{\partial x} - \frac{\partial p}{\partial x} \frac{\partial \rho}{\partial y} \right)$$

$$\frac{\partial \zeta}{\partial t}, u \frac{\partial \zeta}{\partial x}, v \frac{\partial \zeta}{\partial y} \sim \frac{U^2}{L^2} \sim 10^{-10} \text{ s}^{-2}$$

$$w \frac{\partial \zeta}{\partial z} \sim \frac{WU}{HL} \sim 10^{-11} \text{ s}^{-2}$$

$$v \frac{\partial f}{\partial y} \sim U\beta \sim 10^{-10} \text{ s}^{-2}$$

$$\frac{\zeta}{f_0} \leq \frac{U}{f_0 L} \equiv R_o \sim 10^{-1} \rightarrow (\zeta + f) \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \sim f \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \leq \frac{f_0 U}{L} \sim 10^{-9} \text{ s}^{-2}$$

$$\left(\frac{\partial w}{\partial x} \frac{\partial v}{\partial z} - \frac{\partial w}{\partial y} \frac{\partial u}{\partial z} \right) \leq \frac{WU}{HL} \sim 10^{-11} \text{ s}^{-2}$$

$$\frac{1}{\rho^2} \left(\frac{\partial p}{\partial y} \frac{\partial \rho}{\partial x} - \frac{\partial p}{\partial x} \frac{\partial \rho}{\partial y} \right) \leq \frac{\delta p \delta \rho}{\rho^2 L^2} \sim 10^{-11} \text{ s}^{-2}$$

These inequalities appear because the two terms may partially offset one another.

Since both sides of the equation must balance, this term must be smaller, which implies that large scale motions are quasi-nondivergent.

Scaled Vorticity Equation

$$\frac{\partial(\zeta + f)}{\partial t} + u \frac{\partial(\zeta + f)}{\partial x} + v \frac{\partial(\zeta + f)}{\partial y} = -f \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)$$

local
tendency of
absolute
vorticity

horizontal
advection of
absolute
vorticity

effects of
divergence

Implications: For large-scale weather systems, the change of absolute vorticity following the fluid motion is approximately equal to the production of vorticity due to horizontal convergence or the destruction of vorticity due to horizontal divergence. (Note that this does not hold at smaller scales, for which the vertical advection, tilting and baroclinic terms may also be important.)