Fundamental Forces

• To understand atmospheric motions, Newton’s second law requires that we understand the forces that act on a fluid parcel.

• There are two types of forces: body forces and surface forces.

• Body forces act on the center of mass of fluid parcel and have magnitudes proportional to the mass of the parcel.

• Surface forces act across the boundary surface separating a fluid parcel from its surroundings and have magnitudes independent of the mass of the parcel.

• For further reading, see Martin section 2.1.
Pressure Gradient Force

We consider a very small volume element of air

\[ \delta V = \delta x \delta y \delta z \]

that is centered at the point \( (x_0, y_0, z_0) \)
Pressure force exerted on left wall: $F_{Bx}$

Pressure force exerted on right wall: $F_{Ax}$

Net x component of pressure force:
$$F_x = F_{Ax} + F_{Bx}$$

**Method:** Use Taylor expansion to develop a mathematical expression for the pressure force at the center of this fluid element.
Taylor series expansion:

\[ f(x) = f(x_0) + f'(x_0)(x - x_0) + \text{higher order terms} \]

In this case (neglecting higher order terms):

\[ p(x) = p(x_0) + \frac{\partial p}{\partial x}(x - x_0) \]

Therefore, we can express the pressure forces as

\[ F_{Ax} = -\left( p_0 + \frac{\partial p}{\partial x} \frac{\delta x}{2} \right) \delta y \delta z \]

\[ F_{Bx} = +\left( p_0 - \frac{\partial p}{\partial x} \frac{\delta x}{2} \right) \delta y \delta z \]
\[ F_{Ax} = - \left( p_0 + \frac{\partial p}{\partial x} \frac{\delta x}{2} \right) \delta y \delta z \]

\[ F_{Bx} = + \left( p_0 - \frac{\partial p}{\partial x} \frac{\delta x}{2} \right) \delta y \delta z \]

\[ F_x = F_{Ax} + F_{Bx} = - \frac{\partial p}{\partial x} \delta x \delta y \delta z \]

mass = density x volume

\[ m = \rho \delta x \delta y \delta z \]

\[ \frac{F_x}{m} = - \frac{1}{\rho} \frac{\partial p}{\partial x} \]

\[ (x_0, y_0, z_0) \]
\[
\frac{F_x}{m} = -\frac{1}{\rho} \frac{\partial p}{\partial x}
\]

In the same manner, it can be shown that

\[
\frac{F_y}{m} = -\frac{1}{\rho} \frac{\partial p}{\partial y}
\]
\[
\frac{F_z}{m} = -\frac{1}{\rho} \frac{\partial p}{\partial z}
\]

In vector form:

\[
\vec{F} = -\frac{1}{\rho} \nabla p
\]

Note: Pressure force is proportional to gradient of pressure.
Gravitational Force

Newton’s law of universal gravitation states that gravitational force exerted by mass \( M \) on mass \( m \) is:

\[
\vec{F}_g = -\frac{GMm}{r^2} \left( \frac{\vec{r}}{r} \right)
\]
If the earth is taken to be the mass \( M \) and \( m \) is taken to be the mass of a fluid parcel or volume element, then we can write the force per unit mass exerted on the fluid by the earth as

\[
\vec{F}_g \equiv \ddot{g} = -\frac{GM}{r^2} \left( \frac{\vec{r}}{r} \right)
\]

If \( a \) is the radius of the earth and \( z \) is the distance above sea level, then

\[
\ddot{g}^* = \frac{\ddot{g}_0}{(1 + z/a)^2}, \quad \text{where} \quad \ddot{g}_0 = -\left( \frac{GM}{a^2} \right) \left( \frac{\vec{r}}{r} \right)
\]

Because the depths of the atmosphere and ocean are small compared to the radius of the earth \((z << a)\) we can treat the gravitational force per unit mass as a constant.

\[
\ddot{g}^* = \ddot{g}_0
\]
Hydrostatic Balance

In the absence of atmospheric motions, the force due to gravity is exactly balanced by the vertical component of the pressure gradient force, namely:

\[
\frac{\partial p}{\partial z} = -\rho g
\]

The state implied by this relationship is known as **hydrostatic balance**. For large-scale motion in the atmosphere, hydrostatic balance is an excellent approximation for determining the vertical dependence of pressure.
Geopotential

We can define a quantity called the **geopotential**, which is related to gravity, that is:

\[ \Phi(z) - \Phi_0 = \int_0^z g \, dz \]

Assuming the zero of geopotential is zero at mean sea level, the geopotential \( \Phi(z) \) at height \( z \) is the work per unit mass required to lift a parcel to height \( z \) from mean sea level, or equivalently, the potential energy per unit mass. The units of geopotential are \( J \, kg^{-1} \) or \( m^2 s^{-2} \).
Hypsometric Equation

We can rewrite the hydrostatic relationship as:

\[ g \, dz = -\frac{1}{\rho} \, dp = -\alpha \, dp \]

Using the definition of geopotential and the IGL, the above equation becomes:

\[ d\Phi = -\alpha \, dp = -\frac{RT_v}{p} \, dp = -RT_v \, d(\ln p) \]

Upon integrating between two levels (at altitudes \( z_1 \) and \( z_2 \) and pressures \( p_1 \) and \( p_2 \)) yields the hypsometric equation:

\[ \Phi(z_2) - \Phi(z_1) = R \int_{p_2}^{p_1} T_v \, d(\ln p) \]
Hypsometric Equation and Thickness

Defining the **geopotential height** $Z$ as $\Phi/g$, we can rewrite the hypsometric equation as:

$$ \Delta Z \equiv Z_2 - Z_1 = \frac{R}{g} \int_{p_2}^{p_1} T_v \, d(\ln p) $$

Thus, the hypsometric equation relates the vertical distance between two pressure levels, or **thickness**, to the temperature of the intervening layer. The thickness, $\Delta Z$, is proportional to the mean virtual temperature of the layer (weighted by $\ln p$).

Thus, $\Delta Z_A > \Delta Z_B$ thus $\bar{T}_{vA} > \bar{T}_{vB}$

A - warmer

B - colder

850 mb

1000 mb