• Radiative equilibrium
• Some thermodynamics review
• Radiative-convective equilibrium

**Goal:** Develop a 1D description of the [tropical] atmosphere
Vertical temperature profile

Total atmospheric mass: \(~5.15 \times 10^{18}\) kg
Total ocean mass: \(~1.4 \times 10^{21}\) kg
Total earth mass: \(~5.97 \times 10^{24}\) kg
Planck function

Radiance ($R$; units of J m$^{-2}$) as a function of wavelength ($\lambda$; units of m$^{-1}$) and temperature ($T$; units of K):

$$R(\lambda, T) = \frac{2hc^2}{\lambda^5 \left(e^{hc/\lambda k_B T} - 1\right)}$$

$h$ [Planck’s constant] = 6.625 x $10^{-34}$ J s$^{-1}$
$c$ [speed of light] = 3.0 x $10^8$ m s$^{-1}$
$k_B$ [Boltzmann’s constant] = 1.38 x $10^{-23}$ J K$^{-1}$

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Simple radiative balance (with no atmosphere)

- **Stefan Boltzmann equation:** \( F = \sigma T^4 \)
  \[ \sigma = 5.67 \times 10^{-8} \text{ Wm}^{-2}\text{K}^{-4} \]
- **Solar insolation:** \( S_{\text{abs}} = S_0 (1 - \alpha_e) \pi R_e^2 \)
  - Solar flux density: \( S_0 = 1370 \text{ Wm}^{-2} \)
  - Planetary albedo: \( \alpha_e \approx 0.3 \)
  - Earth radius: \( R_e = 6.375 \times 10^6 \text{ m} \)
- Therefore:
  \[
  F = \frac{S_{\text{abs}}}{(4\pi R_e^2)} \Rightarrow T_e = \left[ \frac{S_0 (1 - \alpha_e)}{4\sigma} \right]^{1/4}
  \]
  Here the Earth’s surface area is used.

This yields an effective emission temperature of 255K (-18ºC), compared to an observed surface temperature of 288K (15ºC).
# Mean tropospheric composition

<table>
<thead>
<tr>
<th>Constituent</th>
<th>% by volume of dry air</th>
</tr>
</thead>
<tbody>
<tr>
<td>N₂</td>
<td>78.08</td>
</tr>
<tr>
<td>O₂</td>
<td>20.95</td>
</tr>
<tr>
<td>CO₂</td>
<td>0.033 ↑</td>
</tr>
<tr>
<td>Ar</td>
<td>0.934</td>
</tr>
<tr>
<td>Ne</td>
<td>1.82 x 10⁻³</td>
</tr>
<tr>
<td>He</td>
<td>5.24 x 10⁻⁴</td>
</tr>
<tr>
<td>CH₄</td>
<td>2.0 x 10⁻⁴ ↑</td>
</tr>
<tr>
<td>Kr</td>
<td>1.14 x 10⁻⁴</td>
</tr>
<tr>
<td>N₂O</td>
<td>5.0 x 10⁻⁵ ↑</td>
</tr>
<tr>
<td>H₂</td>
<td>5.0 x 10⁻⁵</td>
</tr>
<tr>
<td>Xe</td>
<td>8.7 x 10⁻⁶</td>
</tr>
<tr>
<td>O₃</td>
<td>4.0 x 10⁻⁶</td>
</tr>
<tr>
<td>H₂O</td>
<td>0.0-4.0 [0.8]</td>
</tr>
</tbody>
</table>

From F.W. Taylor, *Elementary Climate Physics*
Atmospheric absorption

- In the visible, Earth’s atmosphere is effectively transparent [“solar window”]
- At shorter and longer wavelengths, atmosphere is typically strongly absorbing
Simple radiative balance (opaque single layer atmosphere)

\[ S_{\text{abs}} / (4 \pi R_e^2) \Rightarrow T_e \]

\[ T_e = T_a \]

\[ \sigma T_s^4 = \sigma T_a^4 + \sigma T_e^4 \Rightarrow T_s = 2^{1/4} T_e \]

Surface temperature (~303K) is now too large…
Full 1D radiative transfer calculation

- Too hot (cold) close to the surface (near tropopause)
  - Thus, the lapse rate is too large
- Stratospheric temperature close to observations

We’ll revisit this later…but first some thermodynamics.

Emanuel, 2005
Thermodynamics Review

- Entropy & thermodynamic potentials
- Hydrostatic equilibrium & buoyancy
- Stability [dry & moist adiabatic]
1st Law of Thermodynamics & Thermodynamic Identity

\[ du = \delta q - \delta w \]

The notation \( \delta \) denotes an inexact differential, i.e., an integral over the quantity depends on the path taken.

\[ s(u, \{x_i\}) \rightarrow \left( \frac{\partial s}{\partial u} \right)_{\{x_i\}} = \frac{1}{T} \]

\[ ds = \left( \frac{\partial s}{\partial u} \right)_{\{x_i\}} du + \left( \frac{\partial s}{\partial x_i} \right)_{u, \{x_n\}} dx_i \]

\[ du = Tds - T \left( \frac{\partial s}{\partial x_i} \right)_{u, \{x_n\}} dx_i \]

\[ du = Tds + T \left( \frac{\partial s}{d\alpha} \right)_u d\alpha = Tds - pd\alpha = \delta q - \delta w \]

(1) 1st Law ⇔ energy conservation: the change in internal \( du \) must equal the heat added to/removed from the system \( \delta q \) and the work done on/by the system \( \delta w \).

(2) Definition of \( T \) [from specific entropy \( s \), which is written as a function of \( u \) and a set of other variables \( \{x_i\} \) ]

Expanding \( s \) as a linear function of its first partial derivatives...

… and rearranging, using (2)...

… leads to (3) the thermodynamic identity.

Here, pressure \( p \) is related to the partial derivative of \( s \) w.r.t. system specific volume.
Hydrostatic equilibrium & buoyancy

Newton’s 2nd Law of Motion in the z direction:

\[ \vec{F} = m\vec{a} \]

Consider first the case with equal environmental and parcel densities, i.e., \( \rho' = \rho \)

\[ (\rho \delta x \delta y \delta z) \frac{d^2 \delta z}{dt^2} = - (\rho \delta x \delta y \delta z) g + p(z) \delta x \delta y - p(z + \delta z) \delta x \delta y \]

In equilibrium, and applying the limit \( \delta z \to 0 \),

\[ 0 = -\rho g - \frac{\partial p}{\partial z} \]

Consider now \( \rho' \neq \rho \):

\[ (\rho \delta x \delta y \delta z) \frac{d^2 \delta z}{dt^2} = - (\rho \delta x \delta y \delta z) g + p(z) \delta x \delta y - p(z + \delta z) \delta x \delta y \]

\[ \Rightarrow \frac{d^2 \delta z}{dt^2} = -g - \frac{1}{\rho} \frac{\partial p}{\partial z} = g \left( \frac{\rho' - \rho}{\rho} \right) \equiv B \quad \text{Buoyancy} \]
For a dry adiabatic displacement

Consider a parcel undergoing a dry adiabatic displacement from an initial position to a new position separated by a distance $\delta z$. At the new position:

\[
\frac{dw}{dt} = g \frac{\rho' - \rho}{\rho} = g \frac{T - T'}{T'} = g \frac{\theta - \theta'}{\theta'}
\]

\[
= g \left( \frac{\theta}{\theta'} - 1 \right) = g \left( \frac{\theta}{\theta + \delta z \frac{\partial \theta}{\partial z}} - 1 \right)
\]

\[
\approx g \left[ \left( 1 - \frac{1}{\theta} \frac{\partial \theta}{\partial z} \delta z \right) - 1 \right] = -\frac{g}{\theta} \frac{\partial \theta}{\partial z} \delta z
\]

\[
\Rightarrow \frac{d^2 \delta z}{dt^2} + N^2 \delta z \approx 0; \quad N^2 = \frac{g}{\theta} \frac{\partial \theta}{\partial z} \delta z
\]

$N$ is the Brunt-Väisälä frequency.
Dry adiabatic displacement in terms of lapse rates

\[
\frac{dw}{dt} = g \frac{\rho' - \rho}{\rho} = g \frac{T - T'}{T'}
\]

\[
\approx -g \frac{(\Gamma_d - \Gamma)}{T'} \delta z
\]

\[
\frac{d^2 \delta z}{dt^2} + N^2(z) \delta z = 0 \quad N(z) = \sqrt{\frac{g (\Gamma_d - \Gamma)}{T'(z)}}
\]

If \( \Gamma < \Gamma_d \), \( N^2(z) > 0 \), so solutions to the perturbation equation are oscillatory (sinusoids). In this case, the equilibrium is **stable**.

If \( \Gamma > \Gamma_d \), \( N^2(z) < 0 \), so solutions to the perturbation equation are exponential (hyperbolic sine/cosine). In this case, the equilibrium is **unstable**.
Revisiting radiative equilibrium…

• Pure radiative equilibrium is in fact unstable for conditions in the troposphere [but is reasonable in the stratosphere].
  – Need to account for vertical heat transport via convection [more shortly and later]

• $\Gamma_d$ significantly exceeds the observed lapse rate in the atmosphere (~6.5K/km)
  – Modification of the lapse rate by moisture

Emanuel, 2005
Moist variables

IGL for dry air (denoted \(d\)) and water vapor (denoted \(v\)):

\[
\begin{align*}
\rho_d \alpha &= R_d T \\
\epsilon \alpha &= R_v T
\end{align*}
\]

By Dalton’s Law of Partial Pressures, the total pressure of moist air [mixture of dry air\(+\)vapor] is:

\[
p = p_d + \epsilon
\]

The density of moist air is:

\[
\rho = \rho_d + \rho_v = \frac{p}{R_d T} \left(1 - 0.378 \frac{\epsilon}{p}\right)
\]

Virtual temperature \((T_v)\): temperature to which dry air must be raised to have the same density as moist air at the same pressure

\[
T_v = T \left[1 - 0.378 \frac{\epsilon}{p}\right]^{-1}
\]

Specific humidity \((q)\):

\[
q = \frac{\rho_v}{\rho} = \frac{0.622 e}{p - 0.378 e} \approx 0.622 \frac{e}{p}
\]

Using the definitions of \(q\) and \(T_v\), the equation of state of moist air is approximately:

\[
p \approx \rho R_d T (1 + 0.61q)
\]
Saturation vapor pressure

Consider the relative humidity ($rh$): \[ rh = \frac{e}{e_s} \]

Here, the saturation vapor pressure ($e_s$) is the vapor pressure at which condensation occurs; recall that $e_s$ is a monotonic function of temperature.

The *Clausius-Clapeyron* (CC) equation governs the temperature dependence of $e_s$ for two-phase equilibrium (the gas-liquid coexistence line on the phase diagram):

\[
\frac{de_s}{T} = \frac{L}{T(\alpha_2 - \alpha_2)}
\]

If 1 denotes the condensed phase (solid or liquid) and 2 the gaseous phase, $\alpha_2 \gg \alpha_1$. From prior definitions and after integration:

\[ e_s(T) \propto \exp\left[-0.622 \frac{L}{R_d T}\right] \]
Adiabatic processes in a moist atmosphere

Recall that $\theta$ is the temperature a parcel of air would have if displaced, adiabatically and reversibly, to a reference pressure $p_0$ [typically, 1000 mb]:

$$\theta = T \left( \frac{p_0}{p} \right)^\kappa ; \kappa = \frac{R_d}{c_p}$$

[To derive: apply 1st law for an adiabatic process ($ds=0$), use definition of internal energy and IGL, and integrate.]

The (dry) entropy can be expressed in terms of $\theta$:

$$\Delta s_d = c_p \ln \theta \quad \quad d\Delta s_d = 0 \Leftrightarrow d\theta = 0$$

The $\Delta$ notation indicates that specific entropy is defined up to an arbitrary constant.

Now, for an adiabatic upward displacement of saturated air, condensation will occur, leading to a release of latent heat in the amount $-Ldq_s$. The entropy change associated with this release is $d\Delta s_c=(-Ldq_s)/T$. Equating $d\Delta s_d$ and $d\Delta s_c$ and integrating gives the equivalent potential temperature $\theta_e$:

$$c_p d\ln \theta = -\frac{L_c dq_s}{T} \approx -d\left( \frac{L_c q_s}{T} \right) \quad \quad \theta_e = \theta \exp \left[ \frac{Lq_s}{c_p T} \right]$$
Moist Stability

Starting with the vertical acceleration and using the definition of potential temperature, it can be shown that:

\[
\frac{d^2 \delta z}{dt^2} + N^2(z) \delta z = 0 \quad \Rightarrow \quad \frac{d^2 \delta z}{dt^2} + \left( g \frac{\partial \theta}{\partial z} \right) \delta z = 0
\]

Thus:

\[
\frac{\partial \theta}{\partial z} > 0 \quad \text{Stable}
\]

\[
\frac{\partial \theta}{\partial z} < 0 \quad \text{Unstable}
\]

Equivalent potential temperature can be used to evaluate stability for a moist atmosphere; by analogy, a moist adiabatic lapse \( \Gamma_m \) rate can be defined. A region of conditional instability emerges, for \( \Gamma_m < \Gamma < \Gamma_d \). For unsaturated air, this region is stable; for saturated air, it is unstable.
Stability in a moist atmosphere

Stable
- Dry adiabat
- Moist adiabat

Environment at $p'$ warmer than parcel undergoing either dry or moist adiabatic ascent from $p$

Unstable
- Environment at $p'$ cooler than parcel undergoing either dry or moist adiabatic ascent from $p$

Conditionally Unstable
- Environment at $p'$ warmer than parcel undergoing dry adiabatic ascent but cooler than for moist adiabatic ascent
Overview of convection [more later]

• Convection is an important source of atmospheric heating associated with H$_2$O phase change. As water vapor condenses, it releases latent heat.

• The vertical motion associated with convection is an important control on the space-time behavior of H$_2$O vapor, other tracers, aerosols, and clouds.

• *Global* hydrologic cycle balance implies balance between *global* mean precipitation and evaporation rates [~3 mm day$^{-1}$]...although significant regional [sub-global] heterogeneity exists.

• Evaporation is an important component of the surface energy budget.
Convective adjustment

- Eliminates atmospheric column instability
- Environmental conditions that favor instability:
  - **Cooling** aloft via cold air advection or longwave radiative cooling
  - **Warming** of surface via warm air advection or solar heating
  - **Lifting** of air mass through low-level convergence